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LETTER TO THE EDITOR

Large-N eigenvalue distribution of randomly perturbed asymmetric matrices

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Abstract. The density of complex eigenvalues of random asymmetric $N \times N$ matrices is found in the large-N limit. The matrices are of the form $H_0 + A$ where A is a matrix of N^2 independent, identically distributed random variables with zero mean and variance $N^{-1}v^2$. The limiting density $\rho(z, z^*)$ is bounded. The area of the support of $\rho(z, z^*)$ cannot be less than πv^2. In the case of H_0 commuting with its conjugate, $\rho(z, z^*)$ is expressed in terms of the eigenvalue distribution of the non-perturbed part H_0.

Random Hermitian and real symmetric matrices have been studied extensively since the 1950s, when Wigner introduced them into theoretical physics. A lot of results concerning these matrices and techniques for manipulating them are now known. In contrast, random complex and real asymmetric matrices are much less studied, although they have already proved to be useful. We mention here only two examples (but see a discussion in [1]). These are: (i) quantum chaotic scattering and decaying processes, where complex eigenvalues of random non-Hermitian matrices are used to analyse statistical properties of resonances [1–3], and (ii) neural network dynamics where synaptic matrices are, in general, asymmetric and the distribution of their eigenvalues is important for the understanding of network dynamics [4, 5].

In this letter we consider random real asymmetric matrices of the form $H = H_0 + A$. $A = [a_{jk}]_{j,k=1}^{N}$ is a matrix of N^2 independent, identically distributed random variables such that

$$\langle a_{jk}\rangle = 0 \qquad \langle a_{jk}a_{lm}\rangle = N^{-1}v^2\delta_{jl}\delta_{km}. \tag{1}$$

The angle brackets $\langle\cdots\rangle$ denote an average over the random variables a_{jk}. For simplicity we assume that a_{jk} are Gaussian, but our results remain valid for a wider class of distributions. We treat A as a perturbation and H_0 as the non-perturbed part and our aim is to determine the large-N limit of the averaged density of complex eigenvalues of $H_0 + A$.

If A and H_0 are symmetric (or Hermitian) and A obeys the GOE (GUE) statistics, then $H_0 + A$ is known as the deformed GOE (GUE) [6]. In this ensemble eigenvalues are real and, hence, their density is completely determined by the imaginary part of the Green function $G(E + \mathrm{i}0) = \langle N^{-1}\mathrm{tr}\,(E + \mathrm{i}0 - H)^{-1}\rangle$. It appears that in the large-N limit the Green function of the deformed GOE (GUE) is related to that of H_0 by the so-called Pastur equation [7]:

$$G(z) = G_0(z - v^2G(z))\,. \tag{2}$$

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Although this equation cannot be solved explicitly (except in a few cases) it provides useful information about the density of eigenvalues. For instance, one can prove that the density of eigenvalues in the deformed GOE (GUE) is bounded and decays generically as the square root in the vicinity of the spectrum boundaries [8].

Eigenvalues of asymmetric matrices are complex and their average density $\rho(z, z^*)$ is determined by the electrostatic potential

$$\Phi(\kappa, z, z^*) = -N^{-1}\langle \log\det[(zI - H)^*(zI - H) + \kappa^2 I]\rangle$$

by means of Poisson's equation $\rho(z, z^*) = -(1/\pi)\partial^2\Phi(\kappa, z, z^*)/\partial z\partial z^*\big|_{\kappa=0}$ [1, 9]. I is the identity matrix. A positive infinitesimal κ is introduced in order to regularize the potential. Provided $\kappa = 0$, Φ as a function of complex z has a singularity whenever z equals one of the eigenvalues of H.

Anticipating an important role of positive semi-definite matrices $\mathcal{H} = (zI - H)^*(zI - H)$ in studying complex eigenvalues of H, we introduce the following Green function:

$$R(\kappa) = \langle N^{-1}\,\mathrm{tr}(\mathcal{H} + \kappa^2 I)^{-1}\rangle \tag{3}$$

corresponding to $\mathcal{H}$. $R(\kappa)$ as a function of κ is analytic in the right half of the complex plane and obviously determines the density of eigenvalues of $\mathcal{H}$. We show that in the large-N limit the Green functions of $\mathcal{H}$ and $\mathcal{H}_0 = (zI - H_0)^*(zI - H_0)$ are related by equation (10) which can be thought of as a generalization of Pastur's result to the case of positive semi-definite random matrices. In passing, we find derivatives of the electrostatic potential. This allows us to derive an expression for the average density of complex eigenvalues of H, $\rho(z, z^*)$, and for the domain of their distribution. The respective expressions (12)–(14) are given in terms of H_0. Actually, they set up the only restriction to H_0: quantities entering (12)–(14) must be well defined in the large-N limit. We do not specify H_0 further. It can be real or complex and either deterministic or random. In the latter case it is assumed that H_0 is statistically independent of A and it is understood that the average over realizations of H_0 has been taken.

At this point it is worth mentioning that in the specific case of H_0 commuting with its conjugate $H_0{}^*$ (i.e. H_0 can be symmetric, skew-symmetric, Hermitian, skew-Hermitian, etc) $\rho(z, z^*)$ can be *explicitly* expressed in terms of the density of eigenvalues of H_0 (see equations (15)–(17)). This should be compared with the case of deformed GOE (GUE) where only relation (2) between Green functions is known.

Our last remark concerns matrices studied in [1, 3]. They are of the form $\mathrm{i}VV^\top + B$, where V is an $N \times M$ matrix of NM independent Gaussian variables and B obeys the GOE statistics. These random matrices differ from those considered here in that B is symmetric while A is asymmetric. The eigenvalue distribution of $VV^\top$ is known [10] and it seems interesting to recover the results of [1, 3], which were obtained by means of the replica trick [1] and supersymmetry calculations [3], in the framework of our approach. But this problem goes beyond the aim of the present letter.

Introducing the notation $\mathcal{G}(\kappa)$ for the inverse of $\mathcal{H} + \kappa^2 I$ we rewrite the following obvious matrix identity $I = \langle \mathcal{G}(\kappa)(\mathcal{H} + \kappa^2 I)^{-1}\rangle$ as

$$\kappa^2\langle \mathcal{G}(\kappa)\rangle = I - (zI - H_0)^*\langle (zI - H)\mathcal{G}(\kappa)\rangle + \langle A^*(zI - H)\mathcal{G}(\kappa)\rangle\,. \tag{4}$$

$zI - H_0$ is statistically independent of $\mathcal{G}(\kappa)$ but A, which enters the $(zI - H)$ term in the right-hand side of (4), is not. In order to decouple $\langle A\mathcal{G}(\kappa)\rangle$ and $\langle A^*A\mathcal{G}(\kappa)\rangle$ we first notice that each of the entries $\mathcal{G}_{pq}$ of the matrix $\mathcal{G}(\kappa)$ is a function of the Gaussian variable a_{lm}. Therefore

$$\langle a_{lm}\mathcal{G}_{pq}\rangle = \langle a_{lm}^2\rangle\langle \partial\mathcal{G}_{pq}/\partial a_{lm}\rangle = N^{-1}v^2\,\langle \partial\mathcal{G}_{pq}/\partial a_{lm}\rangle\,. \tag{5}$$

This is the only place where a Gaussian distribution of a_{lm} is used. In the non-Gaussian case it can be shown that (5) holds up to the $1/N^2$ order if $a_{lm} = N^{-1/2}\alpha_{lm}$ and the random variables α_{lm} possess several first moments. Straightforward application of (5) and the following rule for differentiating matrix elements of $\mathcal{G}(\kappa)$ with respect to those of A

$$\frac{\partial \mathcal{G}_{pq}}{\partial a_{lm}} = [\mathcal{G}(zI - H)^*]_{pl}\, \mathcal{G}_{mq} + \mathcal{G}_{pm}\, [(zI - H)\mathcal{G}]_{lq} \tag{6}$$

gives

$$\langle (zI - H)\mathcal{G}(\kappa)\rangle = (zI - H_0)\langle \mathcal{G}(\kappa)\rangle - v^2 \langle (zI - H)\mathcal{G}(\kappa) N^{-1} \operatorname{tr} \mathcal{G}(\kappa)\rangle + \mathrm{O}(1/N)\,.$$

One can readily check (6) making use of $\partial \mathcal{G}_{pq}/\partial \mathcal{H}_{km} = -\mathcal{G}_{pk}\, \mathcal{G}_{mq}$ and the chain rule. The normalized trace of $\mathcal{G}(\kappa)$ is a self-averaging extensive quantity [8]. That is, it becomes non-random in the large-N limit: $N^{-1} \operatorname{tr} \mathcal{G}(\kappa) = R(\kappa) + \mathrm{O}(1/N)$, where $R(\kappa) = \langle N^{-1} \operatorname{tr} \mathcal{G}(\kappa)\rangle$. Therefore we conclude that

$$\langle (zI - H)\mathcal{G}(\kappa)\rangle = (zI - H_0)\langle \mathcal{G}(\kappa)\rangle [1 + v^2 R(\kappa)]^{-1} + \mathrm{O}(1/N)\,. \tag{7}$$

Similar reasoning leads to

$$\langle A^*(zI - H)\mathcal{G}(\kappa)\rangle = -v^2\kappa^2 R(\kappa)\langle \mathcal{G}(\kappa)\rangle + \mathrm{O}(1/N)\,. \tag{8}$$

Collecting (4), (7) and (8) we find that in the leading order

$$\langle \mathcal{G}(\kappa)\rangle = \frac{1 + v^2 R(\kappa)}{(zI - H_0)^*(zI - H_0) + \kappa^2 [1 + v^2 R(\kappa)]^2 I}\,. \tag{9}$$

Introducing the notation $\mathcal{G}_0(\kappa)$ for the inverse of $\mathcal{H}_0 + \kappa^2 I$ one can write (9) in the form $\langle \mathcal{G}(\kappa)\rangle = (1 + v^2 R(\kappa)) G_0(\kappa[1 + v^2 R(\kappa)])$. $R(\kappa)$ is to be determined from the self-consistency equation

$$R(\kappa) = [1 + v^2 R(\kappa)] R_0(\kappa[1 + v^2 R(\kappa)]) \tag{10}$$

where $R_0(\kappa) = N^{-1} \operatorname{tr} \mathcal{G}_0(\kappa)$.

Since $-\partial\Phi/\partial z^* = \langle N^{-1} \operatorname{tr}(zI - H)\mathcal{G}(\kappa)\rangle$ one can use (7) and (9) to calculate $\rho(z, z^*)$. Indeed,

$$-\frac{\partial \Phi(\kappa, z, z^*)}{\partial z^*} = N^{-1} \operatorname{tr}(zI - H_0)\mathcal{G}_0(\kappa[1 + v^2 R(\kappa)]) + \mathrm{O}(1/N)\,.$$

Simple analysis of (10) shows that in the leading order $\partial\Phi(\kappa, z, z^*)/\partial z^*\big|_{\kappa=0}$ is given by

$$-\frac{\partial \Phi(\kappa, z, z^*)}{\partial z^*}\bigg|_{\kappa=0} = N^{-1} \operatorname{tr}(zI - H_0)\mathcal{G}_0(\gamma(z, z^*)) \tag{11}$$

where $\gamma(z, z^*) = \lim_{\kappa\to 0+} \kappa[1 + v^2 R(\kappa)]$ is the solution of $R_0(\gamma) = v^{-2}$ if z lies inside the domain D determined by the inequality

$$R_0(0) = N^{-1} \operatorname{tr}[(zI - H_0)^*(zI - H)]^{-1} \geqslant v^{-2} \tag{12}$$

and $\gamma(z, z^*) = 0$ otherwise. Since in the latter case $\partial\Phi/\partial z^*\big|_{\kappa=0}$ does not depend on z we conclude immediately that $\rho(z, z^*) = 0$ outside D. On the other hand, differentiating (11) with respect to z one finds that inside D

$$\rho(z, z^*) = (\pi v^2)^{-1} - \pi^{-1} I(z, z^*) \tag{13}$$

where $I(z, z^*)$ is the large-N limit of

$$N^{-1} \operatorname{tr}(zI - H_0)\mathcal{G}_0(\gamma(z, z^*))(zI - H_0)^*\mathcal{G}_0(\gamma(z, z^*)) \\ - \left| N^{-1} \operatorname{tr}(zI - H_0)\mathcal{G}_0^2(\gamma(z, z^*))\right|^2 \left[N^{-1} \operatorname{tr} \mathcal{G}_0^2(\gamma(z, z^*))\right]^{-1}. \tag{14}$$

For any two matrices P and Q $|\operatorname{tr} PQ^*|^2 \leqslant \operatorname{tr} PP^* \operatorname{tr} QQ^*$. Therefore $I(z, z^*) \geqslant 0$ and $\rho(z, z^*)$ is bounded by $(\pi v^2)^{-1}$. This fact, which is interesting in its own right, also has an important consequence: the area of D, the support of $\rho(z, z^*)$, is not less than the area of a disc with radius v.

In order to illustrate the formulae derived we consider a few examples. If $H_0 = 0$, then the equation $R_0(0) = v^{-2}$, which determines the boundary of D, takes the form $|z|^2 = v^2$ and $I(z, z^*)$ obviously vanishes. Thus, we recover the circular distribution [11–13]: $\rho(z, z^*)$ equals $(\pi v^2)^{-1}$ inside the disc $|z| \leqslant v$ and zero outside.

In our next example H_0 is the Jordan block $J = [h_0 \delta_{j-1,k}]_{j,k=1}^N$, $h_0 > 0$. J has only one eigenvalue $z = 0$ which is defective and highly sensitive to perturbations. On replacing zero in the lower left-hand corner of J by small positive ε, one gets N distinct eigenvalues $h_0(\varepsilon/h_0)^{1/N} \exp(2\pi \mathrm{i} k/N)$. For fixed N the perturbed eigenvalues approach zero as the parameter ε/h_0 vanishes, but the rate of convergence is extremely slow if N is large. For instance, if $N = 50$ one needs $\varepsilon/h_0 \propto 10^{-50}$ in order to confine the eigenvalues to the disc $|z|/h_0 \leqslant 0.1$. Therefore, if not exponentially small, perturbation splits the zero eigenvalue of J into the circle $|z| = h_0$. As can be seen from (12) this phenomenon manifests itself in the large-N limit. Indeed, in the case of $H_0 = J$, equation (12) reduces to $||z|^2 - {h_0}^2| \leqslant v^2$. Therefore if $v < h_0$ the eigenvalues of $J + A$ are distributed in the annulus $1 - v^2/{h_0}^2 \leqslant |z/h_0|^2 \leqslant 1 + v^2/{h_0}^2$ which degenerates into a circle as v vanishes. When $v \geqslant h_0$ the eigenvalues are distributed in the disc $|z/h_0|^2 \leqslant 1 + v^2/{h_0}^2$. In figure 1 we present the results of numerical diagonalization of random matrices $J + A$. As can be seen, the correspondence between the numerical results and our analytical predictions (for $N \to \infty$) is quite good.

If H_0 commutes with its conjugate ${H_0}^*$, our formulae (12)–(14) become simpler. Let us assume that the eigenvalues of H_0 are real. Then the boundary of D is determined by

$$\int \frac{n(\lambda)\,\mathrm{d}\lambda}{|z-\lambda|^2} = \frac{1}{v^2} \tag{15}$$

where $n(\lambda)$ is the density of eigenvalues of H_0. $\rho(z, z^*)$ is given by the same expression

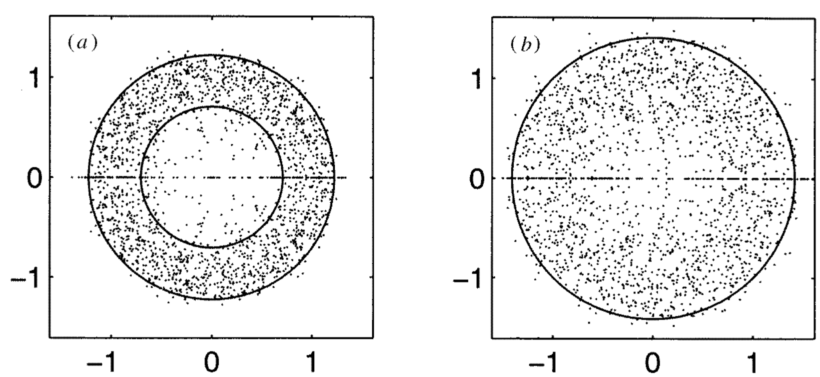

Figure 1. Distribution of numerically computed eigenvalues of the random matrices $J + A$ in the complex plane z/h_0. In each of the plots $N = 50$ and the number of samples is 40. (*a*) $v^2/h_0^2 = 1/2$, (*b*) $v^2/h_0^2 = 1$. The full circles show the boundary of the support of $\rho(z, z^*)$ in the large-N limit.

(13) as before but now $I(z, z^*)$ is

$$\int \frac{|z-\lambda|^2 n(\lambda)\,\mathrm{d}\lambda}{[|z-\lambda|^2+\gamma^2(z,z^*)]^2} - \left|\int \frac{(z-\lambda)n(\lambda)\,\mathrm{d}\lambda}{[|z-\lambda|^2+\gamma^2(z,z^*)]^2}\right|^2 \left[\int \frac{n(\lambda)\,\mathrm{d}\lambda}{[|z-\lambda|^2+\gamma^2(z,z^*)]^2}\right]^{-1} \tag{16}$$

and $\gamma(z, z^*)$ has to be found from

$$\int \frac{n(\lambda)\,\mathrm{d}\lambda}{|z-\lambda|^2+\gamma^2} = \frac{1}{v^2}\,. \tag{17}$$

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